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CONSTRUCTING A MASS-CURRENT RADIATION-REACTION FORCE FOR NUMERICAL SIMULATIONS

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ABSTRACT

We present a new set of 3.5 Post-Newtonian equations in which Newtonian hydrodynamics is coupled to the nonconservative effects of gravitational radiation emission. Our formalism differs in two significant ways from a similar 3.5 Post-Newtonian approach proposed by Blanchet (1993, 1997). Firstly we concentrate only on the radiation-reaction effects produced by a time-varying mass-current quadrupole S_{ij}. Secondly, we adopt a gauge in which the radiation-reaction force densities depend on the fourth time derivative of S_{ij}, rather than on the fifth, as in Blanchet's approach. This difference makes our formalism particularly well-suited to numerical implementation and could prove useful in performing fully numerical simulations of the recently discovered r-mode instability for rotating neutron stars subject to axial perturbations.

Subject headings: relativity: Post-Newtonian approximation — gravitational radiation reaction — stars: neutron — stars: oscillations — instabilities.

1. INTRODUCTION

The recent discovery of an instability in r-mode oscillations in relativistic rotating stars has generated widespread interest. r-mode oscillations in Newtonian rotating stars have been widely investigated in the past (see, for instance, Papaloizou and Pringle 1978; Provost et al. 1981; Saio 1982), but the evidence that they are indeed unstable to the emission of gravitational radiation is rather recent. The first numerical calculations carried out by Anderson (1998) and confirmed analytically by Friedman and Morsink (1998) have spawned a growing literature on the subject (Andersson, Kokkotas and Schutz 1998; Kojima 1998; Kokkotas and Stergioulas 1998; Levin 1998; Lindblom and Ipser 1999; Lindblom, Mendell and Owen 1999; Lindblom, Owen and Morsink 1998; Lockitch and Friedman 1998; Madsen 1998; Owen et al. 1998). Much of the interest in r-mode oscillations is related to the fact that their existence is not dependent on a specific rate of rotation; these modes are, in fact, unstable for arbitrarily slowly-rotating, perfect fluid stars. This result represents a significant difference from previously investigated, rotation-induced instabilities, as for instance the bar-mode instability, which requires a minimum rotation rate of the star (Chandrasekhar 1970; Friedman and Schutz 1978; Lindblom and Mendell 1995; Stergioulas and Friedman 1998). Consequently, the r-mode instability may have a more pervasive effect.

The r-mode instability is a purely relativistic effect, triggered by the emission of gravitational radiation and can be explained in terms of the basic Chandrasekhar-Friedman-Schutz instability mechanism (Chandrasekhar 1970; Friedman and Schutz 1978). Since gravitational radiation removes positive angular momentum from a prograde mode (i.e. a mode that in an inertial frame is seen as moving in the same positive φ direction as the star), it will also extract positive angular momentum from any perturbation which (as a result of the star's rotation) is prograde in the inertial, but retrograde in the corotating frame. Such a mode has, in the corotating frame, negative angular momentum (the perturbed fluid does not rotate as fast as it did without the perturbation) and by making its angular momentum increasingly negative, gravitational radiation drives an instability (Friedman 1998). A significant difference between the axial r-mode instability and the previously known gravitational radiation driven bar-mode instability, is that gravitational radiation couples with r-mode oscillations primarily through time-varying mass-current multipole moments rather than through the usual time-varying mass multipole moments. We here present a new set of Post-Newtonian (PN) hydrodynamical equations which make the calculation of radiation-reaction forces due to mass-current multipole moments numerically feasible.

There are a number of reasons why a numerical investigation of the onset, growth and saturation of the r-mode oscillations is of great interest. All investigations to date have been based on perturbation analyses usually truncated at

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the lowest order in the expansion parameter, namely the star's angular velocity (Andersson 1998; Friedman and Morsink 1998; but see Andersson, Kokkotas and Schutz 1998; Kokkotas and Stergioulas 1998; Lindblom, Mendell and Owen 1999 for a treatment at higher order). Within this framework, the perturbation is parameterized in terms of the angular velocity of the star and its growth is heuristically followed in terms of the energy and angular momentum losses produced by the gravitational radiation emission (Owen et al. 1998). The relevant timescales for the growth and the subsequent viscous decays of the instability are also estimated through perturbation analysis (Lindblom, Owen and Morsink 1998). Although neglecting the presence of a magnetic field, these approaches have clarified the basic features of the instability and provided the first estimates of the importance of the instability in extracting angular momentum from hot young neutron stars, thus setting an upper limit on their angular velocity (Lindblom, Owen and Morsink 1998; Andersson, Kokkotas and Schutz 1998). They also provided qualitative and quantitative information about the expected gravitational waveforms (Owen et al. 1998). However, they are not capable of describing the nonlinear development of the instability or identifying its point of saturation. For these features, numerical simulations are required.

To study this mode numerically, it will be useful and adequate to follow the conservative Eulerian hydrodynamics via Newtonian equations and all the nonconservative effects due to (mass-current multipoles) gravitational radiation emission via a Post-Newtonian radiation-reaction potential at the 3.5 order [this is what we shall refer to as $(0 + 3.5)$ PN]. Such a $(0+3.5)$ PN approach has the advantage of capturing of all the relevant nonconservative general relativistic effects without having to resort to a more complicated relativistic hydrodynamics treatment. Moreover, a $(0+3.5)$ PN treatment allows us to clearly disentangle the different sources of gravitational radiation. As a result, in a simulation of the r-mode instability, we can selectively neglect all the dissipative contributions coming from mass multipole moments and concentrate solely on the mass-current quadrupole moment, which we expect to be the dominant mass-current multipole moment. Disentangling these modes is not possible in a full general relativistic treatment. Finally, because simulations of the onset and growth of the r-mode instability also require the numerical evaluation of stable configurations on growth timescales much longer than the dynamical timescales (i.e. growth timescales $\gg$ rotation period), we also expect that a three-dimensional, fully relativistic simulation may be, at present, beyond reach. On the other hand, a radiation-reaction formalism, though approximate, allows us to use a scaling in order to artificially accelerate the onset of an instability without changing the underlying physical evolution.

The organization of the paper is as follows: in Section 2 we summarize the basic steps of a PN expansion and the hydrodynamical equations that emerge from it. Particular attention will be paid to the radiation-reaction forces and to the losses they produce in the energy and angular momentum of the system. In the following Sections 3 and 4 we discuss two explicit expressions for the radiation-reaction force due to time-varying mass current quadrupole moments. The first one was obtained by Blanchet (1997) (Section 3), while the second is derived in a new gauge (Section 4) which makes it better suited for numerical implementation. Section 4 also contains detailed verifications that the new force yields the required rates of energy and angular momentum loss. Section 5 synthesizes the main results derived in the previous Sections and, for the benefit of the reader interested in numerical implementation, presents the final form of the hydrodynamic equations and radiation-reaction terms. Having in mind the study of the r-mode instability, we present in Section 6 a useful rescaling of the radiation-reaction term which will accelerate the growth-time of the instability to a timescale set by numerical constraints. Section 7 summarizes our conclusions and the prospects for numerical computations exploiting the formalism introduced here. Throughout the paper we will adopt Cartesian coordinates. G and c denote the gravitational constant and the speed of light. Greek indices run from 0 to 3, Latin indices from 1 to 3, and we use Einstein summation convention on matched indices.

2. PN EXPANSION: THE BASIC EQUATIONS

In this Section we briefly summarize the standard approach to perform a PN expansion for the equations of relativistic hydrodynamics. (A recent discussion of the foundations and applications of the PN approximation has been given by Asada and Futamase 1997). We adopt the standard $3 + 1$ splitting of spacetime and write the line element in the form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -(\alpha^2 - \beta_i\beta^i)c^2dt^2 + 2\beta_i c dt dx^i + \gamma_{ij}dx^i dx^j \ , \tag{1}$$

where α, β^i, and γ_{ij} are the lapse function, the shift vector, and spatial 3-metric, respectively, while $\beta_i = \gamma_{ij}\beta^j$.

We also consider a perfect fluid, whose energy-momentum tensor is

$$T^{\mu\nu} = (\rho c^2 + \rho\varepsilon + P)u^\mu u^\nu + Pg^{\mu\nu} \ , \tag{2}$$

where ρ is the rest-mass density, ε is specific internal energy, P the pressure, and u^μ the fluid four-velocity. It is also convenient to introduce a coordinate velocity v^i, defined as

$$\frac{v^i}{c} \equiv \frac{u^i}{u^0} = -\beta^i + \frac{\gamma^{ij}u_j}{u^0} \ . \tag{3}$$

Imposing the conservation of rest-mass, energy and momentum yields the standard relativistic hydrodynamic equations,

$$\frac{\partial \rho_*}{\partial t} + \frac{\partial(\rho_* v^i)}{\partial x^i} = 0 \ , \tag{4}$$

$$\frac{\partial(\rho_* \varepsilon)}{\partial t} + \frac{\partial(\rho_* \varepsilon v^i)}{\partial x^i} = -P\left[\frac{\partial(\alpha u^0 \gamma^{1/2})}{\partial t} + \frac{\partial(\alpha u^0 \gamma^{1/2} v^i)}{\partial x^i}\right] , \tag{5}$$

$$\frac{\partial(\rho_* h u_k)}{\partial t} + \frac{\partial(\rho_* h u_k v^i)}{\partial x^i} = -\frac{\alpha \gamma^{1/2}}{c}\frac{\partial P}{\partial x^k} + \rho_* h c\left[-\alpha u^0 \frac{\partial \alpha}{\partial x^k} + u_j \frac{\partial \beta^j}{\partial x^k} - \frac{u_i u_j}{2u^0}\frac{\partial \gamma^{ij}}{\partial x^k}\right] , \tag{6}$$

where

$$u^0 = \frac{\left(1 + \gamma^{ij} u_i u_j\right)^{1/2}}{\alpha} \ , \tag{7a}$$

$$\rho_* \equiv \rho \alpha u^0 \gamma^{1/2} \ , \qquad h = 1 + \frac{1}{c^2}\left(\varepsilon + \frac{P}{\rho}\right) , \tag{7b}$$

and $\gamma = \det(\gamma_{ij})$. Equations (4)–(6) will also be referred to as the continuity, energy and Euler equations, respectively.

We next proceed to the PN approximation and perform a series expansion of the metric in the inverse powers of c up to the 3.5 PN order (Chandrasekhar 1965; Chandrasekhar and Esposito 1969; Asada, Shibata and Futamase 1996):

$$\alpha = 1 + \frac{1}{c^2}\phi + \frac{1}{c^4}{}_4\alpha + \frac{1}{c^5}{}_5\alpha + \frac{1}{c^6}{}_6\alpha + \frac{1}{c^7}{}_7\alpha + \frac{1}{c^8}{}_8\alpha + \frac{1}{c^9}{}_9\alpha + O(c^{-10}) \ , \tag{8a}$$

$$\beta^i = \frac{1}{c^3}{}_3\beta^i + \frac{1}{c^5}{}_5\beta^i + \frac{1}{c^6}{}_6\beta^i + \frac{1}{c^7}{}_7\beta^i + \frac{1}{c^8}{}_8\beta^i + O(c^{-9}) \ , \tag{8b}$$

$$\gamma_{ij} = \delta_{ij}\left(1 - \frac{2}{c^2}\phi\right) + \frac{1}{c^4}{}_4h_{ij} + \frac{1}{c^5}{}_5h_{ij} + \frac{1}{c^6}{}_6h_{ij} + \frac{1}{c^7}{}_7h_{ij} + O(c^{-8}) \ , \tag{8c}$$

where the left subscript n indicates the coefficient of the $O(c^{-n})$ term in the series expansion, and ϕ is the Newtonian potential. We implicitly adopt the usual PN gauge in which ${}_2\alpha = \phi$ and ${}_2h_{ij} = -2\phi\delta_{ij}$; (Chandrasekhar 1965; Chandrasekhar and Esposito 1969) and ${}_5\alpha$ is a function of time only. Similarly, the four-velocity is also expanded in terms of v^i, and full expressions for this are given by Chandrasekhar (1965), Chandrasekhar and Esposito (1969), Asada, Shibata, and Futamase (1996).

Using (8), the 3.5 PN expression of the Euler equation (6) can then be written as

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} - \frac{\partial \phi}{\partial x_i} + \frac{1}{\rho}\left(\frac{1}{c^2}F_i^{1\mathrm{PN}} + \frac{1}{c^4}F_i^{2\mathrm{PN}} + \frac{1}{c^5}F_i^{2.5\mathrm{PN}} + \frac{1}{c^6}F_i^{3\mathrm{PN}} + \frac{1}{c^7}F_i^{3.5\mathrm{PN}}\right) + O(c^{-8}) \ . \tag{9}$$

Hereafter we neglect the higher order difference between contravariant and covariant components and use the latter only.

It is convenient to distinguish, in the right hand side of (9), the terms related to dissipative radiation-reaction effects from those arising from conservative hydrodynamical stresses. In particular, we define the radiation-reaction force density[3]

$$\mathbf{F}^{\mathrm{rr}} \equiv \mathbf{F}^{2.5\mathrm{PN}} + \mathbf{F}^{3.5\mathrm{PN}} = \mathbf{F}^{\mathrm{rrm}} + \mathbf{F}^{\mathrm{rrc}} \ , \tag{10}$$

where $\mathbf{F}^{\mathrm{rrm}}$ and $\mathbf{F}^{\mathrm{rrc}}$ refer to the radiation-reaction force due to time-varying mass multipole moments and mass-current multipole moments, respectively. The general (slice-independent) expressions for $F_i^{2.5\mathrm{PN}}$ and $F_i^{3.5\mathrm{PN}}$ are given by (Asada and Futamase 1997; Blanchet 1997)

$$\begin{aligned} \rho^{-1}F_i^{2.5\mathrm{PN}} = & -\partial_i({}_7\alpha) - \partial_t({}_6\beta_i) + v_j\partial_i({}_6\beta_j) - v_j\partial_j({}_6\beta_i) \\ & -\partial_t({}_5h_{ij}v_j) - {}_5h_{ij}v_k\partial_k v_j \end{aligned} \tag{11}$$

$$\begin{aligned} \rho^{-1}F_i^{3.5\mathrm{PN}} = & -\partial_i({}_9\alpha) - \partial_t({}_8\beta_i) - \partial_j({}_8\beta_i)v_j + \partial_i({}_8\beta_j)v_j \\ & -\partial_t({}_7h_{ij}v_j) - v_k\partial_k({}_7h_{ij}v_j) + \frac{1}{2}v_j v_k \partial_i({}_7h_{jk}) + \delta F_i^{3.5\mathrm{PN}}({}_7\alpha, \ {}_6\beta_i, \ {}_5h_{ij}) \ , \end{aligned} \tag{12}$$

where $\partial_i = \partial/\partial x_i$, $\partial_t = \partial/\partial t$, and in deriving (11) we have exploited that $\partial_i[{}_5\alpha(t)] = 0$. Note that while $\mathbf{F}^{2.5\mathrm{PN}}$ is dependent *only* on the time-varying *mass quadrupole* moments , $\mathbf{F}^{3.5\mathrm{PN}}$ is, in general, dependent on time-varying *mass quadrupole* moments, *mass octupole* moments, as well as on time-varying *mass-current quadrupole* moments. In

[3] Hereafter we refer to the radiation-reaction force densities simply as radiation-reaction forces.

particular, in equation (12), we have symbolically indicated by $\delta\mathbf{F}^{3.5\mathrm{PN}}({}_7\alpha,\ {}_6\beta_i,\ {}_5h_{ij}v_j)$ all of the contributions coming from mass quadrupole moments.

The work done and the torque produced by the radiation-reaction forces $\mathbf{F}^{\mathrm{rr}}$ must balance the energy and angular momentum carried off to infinity by the gravitational waves. The emission rate of gravitational waves is known from the multipolar decomposition of the radiation field (Thorne 1980). In the absence of any dissipative mechanism other that gravitational wave emission, the energy and angular momentum loss rates can be readily calculated as

$$\frac{dE}{dt} = \int d^3\mathbf{x}\ v_i F_i^{\mathrm{rr}}\ , \tag{13}$$

$$\frac{dS_i}{dt} = \int d^3\mathbf{x}\ \epsilon_{ijk}x_jF_k^{\mathrm{rr}}\ , \tag{14}$$

where $d/dt = \partial_t + v_i\partial_i$ and ϵ_{ijk} is the Levi-Civita symbol. The total energy and angular momentum have the usual Newtonian definitions,

$$E \equiv \int d^3\mathbf{x}\ \rho\Big(\frac{1}{2}v^2 + \frac{1}{2}\phi + \varepsilon\Big)\ , \tag{15}$$

$$S_i \equiv \int d^3\mathbf{x}\ \epsilon_{ijk}x_j\rho v_k\ , \tag{16}$$

with $v^2 \equiv v_kv_k$. Equations (13) and (14) will be used repeatedly in this paper to verify the correctness of the derived expressions for the radiation-reaction forces.

So far, our discussion of radiation-reaction forces has been general and we have not restricted ourselves to a specific physical configuration and considered a generic gauge in which the expansion (8) is valid. Hereafter, however, we will concentrate on a PN formulation of the radiation-reaction forces which may prove useful in a numerical study of the r-mode instability. Because the r-mode instability is predominately excited by mass-current quadrupole moments (cf. Section 1), we will neglect any contribution to the radiation-reaction forces coming from mass multipole moments and consider $\mathbf{F}^{\mathrm{rrm}} = 0 = \delta\mathbf{F}^{3.5\mathrm{PN}}$. Even with this restriction, the numerical computation of $\mathbf{F}^{\mathrm{rrc}}$ in presently adopted gauges (Burke 1971, Blanchet 1997) is nontrivial. In the following two Sections we discuss these complications and offer a way to simplify them.

3. MASS-CURRENT QUADRUPOLE MOMENT RADIATION-REACTION: BLANCHET'S GAUGE

The first expression for the radiation-reaction force due to time-varying mass-current quadrupole moments was derived by Burke using a matched asympotics expansion (Burke 1969, 1970, 1971) and expressed in terms of vector spherical harmonics. His expression, however, does not yield the required energy and angular momentum losses (see Walker and Will 1980 for an explanation of the error in the formulation). More recently, a new complete treatment of the radiation-reaction and balance equations at 3.5 PN order has been provided by Blanchet (1997) as an extension of earlier work on gravitational radiation-reaction forces (Blanchet 1993). Here we briefly review the key steps necessary for our modified treatment presented in the next Section. Firstly, the "canonical form" of the linearized metric $\bar{h}^{\mu\nu}_{(1)}$ is constructed in the harmonic gauge (Thorne 1980). This linearized metric expresses the linear deviation from the Minkowski metric $\eta^{\mu\nu}$ in a series expansion in G [i.e. $\bar{h}^{\mu\nu} \equiv \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu} = \bar{h}^{\mu\nu}_{(1)} + \bar{h}^{\mu\nu}_{(2)} + O(G^3)$]. Then, $\bar{h}^{\mu\nu}_{(1)}$ can be rewritten in terms of two (and only two) sets of time-varying multipole moments, referred to as the "mass-type" and "current-type" moments (Thorne 1980; see Appendix B for details). The contributions to the linearized $\bar{h}^{\mu\nu}_{(1)}$ coming from radiation-reaction effects are then derived by taking the half-sum and the half-difference of the retarded and advanced expressions of the multipole moments, and by studying the nonlinear corrections by means of a Post-Minkowskian method (Blanchet 1993, 1997). In doing this, an infinitesimal gauge transformation to the "generalized Burke-Thorne gauge" (hereafter, we will refer to it simply as *Blanchet's gauge*) is performed in order to obtain $h^{\mu\nu}$, a simplified form of the metric (Blanchet 1993, 1997).

Restricting our attention *only* to the radiation-reaction force produced by a time-varying mass-current quadrupole moment, it is clear from (12) that we need explicit expressions for the metric coefficients ${}_9\alpha$, ${}_8\beta_k$ and ${}_7h_{ij}$. While the last two are known already from the linear term of $\bar{h}^{\mu\nu}$, the first one needs to be obtained through an iteration involving also the nonlinear terms. As a result of this iteration, the relevant parts of expanded metric functions are [cf. equations (3.6) of Blanchet 1997]

$${}_9\alpha = 0\ , \tag{17a}$$

$${}_8\beta_i = \frac{16\,G}{45}\epsilon_{ijk}x_jx_lS^{(5)}_{kl}\ , \tag{17b}$$

$${}_7h_{ij} = 0\ , \tag{17c}$$

where S_{ij} is the Newtonian mass-current quadrupole moment defined as

$$S_{ij}(t) \equiv \int d^3\mathbf{x}\, \epsilon_{kl(i}x_{j)}x_k \rho v_l \ . \tag{18}$$

It is useful to remark that S_{ij} is trace-free, i.e. $S_{ij}\delta^{ij} = 0$. The right superscript (n) indicates the n-th total time derivative:

$$A^{(n)}(t) \equiv \left(\frac{d}{dt}\right)^n A(t) \ , \tag{19}$$

and $A_{(ij)} = \frac{1}{2}(A_{ij} + A_{ji})$.

Using (12), the contribution to the 3.5 PN radiation-reaction force due to a time-varying mass-current quadrupole moment in Blanchet's gauge is given by

$$\rho^{-1}F_i^{\rm rrc} = \frac{16}{45}G\left(2v_j\epsilon_{jil}x_m S^{(5)}_{lm} + v_j\epsilon_{jkl}x_k S^{(5)}_{li} - v_j\epsilon_{ikl}x_k S^{(5)}_{lj} - \epsilon_{ikl}x_k x_m S^{(6)}_{lm}\right) \ . \tag{20}$$

The validity of expression (20) can be verified by computing the energy and angular momentum dissipation rates. In particular, inserting (20) into (13), we immediately obtain

$$\begin{aligned}\frac{dE}{dt} &= \frac{16}{45}\frac{G}{c^7}S_{ij}S^{(6)}_{ij} \\ &= -\frac{16}{45}\frac{G}{c^7}S^{(3)}_{ij}S^{(3)}_{ij} + \frac{16}{45}\frac{G}{c^7}\frac{d}{dt}\left(S_{ij}S^{(5)}_{ij} - S^{(1)}_{ij}S^{(4)}_{ij} + S^{(2)}_{ij}S^{(3)}_{ij}\right).\end{aligned} \tag{21}$$

Assuming nearly-periodic motion of the matter field, and averaging over several periods, we can discard the total time derivative term and obtain the standard formula of the energy loss due to mass-current quadrupole radiation (Thorne 1980):

$$\left\langle \frac{dE}{dt} \right\rangle = -\frac{16}{45}\frac{G}{c^7}\langle S^{(3)}_{ij}S^{(3)}_{ij}\rangle \ , \tag{22}$$

where, as usual,

$$\langle A\rangle \equiv \frac{1}{T}\int_0^T A(t)dt \ . \tag{23}$$

Similarly, by using expression (20) in equation (14), we obtain

$$\begin{aligned}\frac{dS_i}{dt} &= -\frac{16}{45}\frac{G}{c^7}\int d^3\mathbf{x}\,\rho\Big[(x_ix_jx_kS^{(6)}_{jk} - |\mathbf{x}|^2x_jS^{(6)}_{ij}) + 2(x_jx_kv_iS^{(5)}_{jk} - x_jx_kv_kS^{(5)}_{ij}) \\ &\qquad\qquad + (x_ix_jv_kS^{(5)}_{jk} - |\mathbf{x}|^2v_jS^{(5)}_{ij}) - \epsilon_{lmn}\epsilon_{ijk}x_jx_mv_lS^{(5)}_{nk}\Big] \\ &= -\frac{16}{45}\frac{G}{c^7}\Big\{\int d^3\mathbf{x}\,\rho\Big[(x_jx_kv_i - x_ix_jv_k)S^{(5)}_{jk} - \epsilon_{ijk}\epsilon_{lmn}x_mx_jv_lS^{(5)}_{nk}\Big] \\ &\qquad\qquad + \frac{d}{dt}\Big[\int d^3\mathbf{x}\,\rho(x_ix_jx_kS^{(5)}_{jk} - |\mathbf{x}|^2x_jS^{(5)}_{ij})\Big]\Big\} \\ &= -\frac{32}{45}\frac{G}{c^7}\epsilon_{ijk}S_{jl}S^{(5)}_{kl} - \frac{16}{45}\frac{G}{c^7}\frac{d}{dt}\int d^3\mathbf{x}\,\rho(x_ix_jx_kS^{(5)}_{jk} - |\mathbf{x}|^2x_jS^{(5)}_{ij}),\end{aligned} \tag{24}$$

where $|\mathbf{x}|^2 \equiv x_ix_i$. After averaging (24) over several periods (and taking two integrations by parts), the formula for the angular momentum loss is

$$\left\langle \frac{dS_i}{dt} \right\rangle = -\frac{32}{45}\frac{G}{c^7}\epsilon_{ijk}\langle S^{(2)}_{jl}S^{(3)}_{kl}\rangle \ , \tag{25}$$

which is again in agreement with the required expression (Thorne 1980).

Although Blanchet's formalism is clear and complete [indeed Blanchet (1993) and (1997), also discusses 3.5 PN radiation-reaction potentials due to mass quadrupole and octupole moments], it is not particularly simple for numerical implementation. The reason for this is evident from expression (20), in which the radiation-reaction force depends on a high time derivative of the mass-current quadrupole moment S_{ij}. It is often possible, and highly convenient in a numerical calculation, to replace some of the time derivatives of the mass and mass-current multipole moments by quadratures. This

involves combining the continuity and Euler equations and introducing some supplementary variables (most notably the partial time derivatives of the Newtonian gravitational potential) which satisfy elliptic equations (Nakamura and Oohara 1989). Here, however, obtaining these quadratures is realistic for $S^{(3)}_{ij}$ at most (see Appendix A for a discussion), leaving the higher order time derivatives to be obtained via finite differencing of $S^{(3)}_{ij}$. The latter operation can be extremely inaccurate, even for a numerical scheme which is second order accurate in time and space, and might introduce numerical instabilities.

To overcome this difficulty, we employ a different gauge condition and derive an alternative form of the metric in the next Section. This alternative leads to radiation-reaction forces which are dependent only on the fourth time derivative of S_{ij}, a considerable improvement computationally.

4. MASS-CURRENT QUADRUPOLE RADIATION-REACTION: A NEW GAUGE

In this Section we adopt a gauge choice different from Blanchet's to obtain a more desirable form for the radiation-reaction force. As in Section (3), we start with the linear metric in the canonical form $\bar{h}^{\mu\nu}_{(1)}$ (Thorne 1980; Blanchet 1993) and define $\tilde{h}^{\mu\nu}_{(1)} \equiv \bar{h}^{\mu\nu}_{(1)} - \frac{1}{2}\eta^{\mu\nu}\bar{h}_{(1)}$, where $\bar{h}_{(1)} \equiv (\bar{h}_{(1)})^{\mu}{}_{\mu}$. In this gauge, the metric coefficients ${}_8\tilde{\beta}_k$ and ${}_7\tilde{h}_{ij}$ are (see Appendix B for details)

$$ {}_8\tilde{\beta}_i = \tilde{h}^{0i}_{(1)} = -\frac{4\,G}{45}\epsilon_{ijk}x_jx_lS^{(5)}_{kl}, \tag{26a} $$

$$ {}_7\tilde{h}_{ij} = -\tilde{h}^{ij}_{(1)} = -\frac{8\,G}{9}x_k\epsilon_{kl(i}S^{(4)}_{j)l}\ . \tag{26b} $$

To eliminate the dependence on the fifth time derivative of S_{ij}, we perform an infinitesimal gauge transformation

$$ h_{\mu\nu} = \tilde{h}_{\mu\nu} + \partial_\nu\xi_\mu + \partial_\mu\xi_\nu\ , \tag{27} $$

where $\tilde{h}^{\mu\nu} \equiv \bar{h}^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\bar{h}$, with $\bar{h} \equiv \bar{h}^{\mu}{}_{\mu}$. We use the freedom in the gauge transformation to set ${}_8\beta_k$ to zero by choosing

$$ \xi_0 = 0\ , \tag{28a} $$

$$ \xi_i = \frac{4\,G}{45}\epsilon_{ijk}x_jx_lS^{(4)}_{kl}\ , \tag{28b} $$

which then yields

$$ {}_8\beta_i = 0\ , \tag{29a} $$

$$ {}_7h_{ij} = -\frac{32\,G}{45}x_k\epsilon_{kl(i}S^{(4)}_{j)l}\ . \tag{29b} $$

We still have not determined ${}_9\alpha$, but this can be done by choosing a time-slice condition. Note, however, that selecting a specific form for the shift and the spatial 3-metric through equations (29) restricts the set of possible choices for ${}_9\alpha$ (see Appendix C for a discussion). In particular, we impose the maximal slicing condition, for which the trace of the extrinsic curvature tensor is set to zero (Arnowitt, Deser and Misner 1962; Smarr and York 1978; Schäfer 1983; Blanchet, Damour and Schäfer 1990) and which is compatible with conditions (29). This results in a linear elliptic equation for the lapse function, whose 3.5 PN approximation is (see Appendix C for details)

$$ \Delta({}_9\alpha) = \partial_i\left({}_7h_{ij}\partial_j\phi\right)\ , \tag{30} $$

where Δ denotes the flat spatial Laplacian. Introducing a scalar "superpotential" χ satisfying (Chandrasekhar 1969)

$$ \Delta\chi = 2\phi\ , \tag{31} $$

and using the fact that $\partial_i({}_7h_{ij}) = 0 = \Delta({}_7h_{ij})$, we then obtain

$$ {}_9\alpha = \frac{1}{2}\left({}_7h_{ij}\partial_{ij}\chi\right)\ . \tag{32} $$

The expression for ${}_9\alpha$ can be further simplified by solving (31) for χ

$$ \chi(\mathbf{x}) \equiv -G\int d^3\mathbf{x}'\ \rho(\mathbf{x}')|\mathbf{x}-\mathbf{x}'|\ . \tag{33} $$

When (33) is inserted in equation (32), it leads to

$$
\begin{aligned}
{}_9\alpha &= -\frac{1}{2}G({}_7h_{ij})\left[x_i\frac{\partial}{\partial x_j}\int d^3\mathbf{x}'\ \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + \delta_{ij}\int d^3\mathbf{x}'\ \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} - \frac{\partial}{\partial x_j}\int d^3\mathbf{x}'\ \frac{\rho(\mathbf{x}')x'_i}{|\mathbf{x}-\mathbf{x}'|}\right] \\
&= \frac{1}{2}({}_7h_{ij})[x_i\partial_j\phi + \partial_j P_i]\ ,
\end{aligned} \tag{34}
$$

where ${}_7h_{ij}\delta_{ij} = 0$. The vector potential $\mathbf{P}$ in (32) is defined by

$$
\mathbf{P}(\mathbf{x}) \equiv G\int d^3\mathbf{x}'\ \rho(\mathbf{x}')\frac{\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\ , \tag{35}
$$

and can be most easily calculated by solving the linear elliptic equations

$$
\Delta P_i = -4\pi G\rho x_i\ . \tag{36}
$$

While solving equations (36) represents an additional computation, nonexistent in Blanchet's formulation, this integration is generally not too taxing in a numerical hydrodynamical simulation which already must solve Poisson's equation for the Newtonian gravitational potential (Nakamura and Oohara 1989; Oohara and Nakamura 1990, 1991; Shibata, Nakamura and Oohara 1992; Ruffert, Janka and Schäfer 1996).

Finally, using expressions (34) and (29b) for ${}_9\alpha$ and ${}_7h_{ij}$ in (12), we obtain the new gauge expression for the radiation-reaction force due to a time-varying mass-current quadrupole moment

$$
\rho^{-1}F_i^{\mathrm{rrc}} = -\partial_i({}_9\alpha) - \partial_t({}_7h_{ij}v_j) - v_k\partial_k({}_7h_{ij}v_j) + \frac{1}{2}v_jv_k\partial_i({}_7h_{jk})\ . \tag{37}
$$

In subsections 4.1 and 4.2, we verify (37) by computing the energy and angular momentum loss rates. The reader wishing to omit this discussion may proceed directly to Section 5.

4.1. *Rate of Energy Loss*

We calculate the rate at which the total energy of the system is lost to radiation, by substituting expression (37) into equation (13). The relevant integrals that emerge are

$$
\begin{aligned}
\int d^3\mathbf{x}\ \rho v_i\partial_i({}_9\alpha) &= -\int d^3\mathbf{x}\ \partial_i(\rho v_i){}_9\alpha = \int d^3\mathbf{x}\ \partial_t(\rho){}_9\alpha \\
&= -\frac{16}{45}G\epsilon_{ijk}S_{kl}^{(4)}\frac{d}{dt}\iint d^3\mathbf{x}\ d^3\mathbf{x}'\ \rho(\mathbf{x})\rho(\mathbf{x}')\frac{x_ix_lx'_j}{|\mathbf{x}-\mathbf{x}'|^3}\ ,
\end{aligned} \tag{38}
$$

$$
\int d^3\mathbf{x}\ \rho v_iv_j\partial_t({}_7h_{ij}) = \frac{32}{45}G\left(S_{ij}^{(1)}S_{ij}^{(5)} - S_{ij}^{(5)}\epsilon_{kli}\iint d^3\mathbf{x}\ d^3\mathbf{x}'\ \rho(\mathbf{x})\rho(\mathbf{x}')\frac{x_jx_kx'_l}{|\mathbf{x}-\mathbf{x}'|^3}\right)\ , \tag{39}
$$

$$
\begin{aligned}
\int d^3\mathbf{x}\ \rho v_i\left[{}_7h_{ij}\left(\partial_tv_j + v_k\partial_kv_j\right)\right] &= -\frac{16}{45}G\epsilon_{ikl}S_{lj}^{(4)}\int d^3\mathbf{x}\ \rho x_k\frac{d(v_iv_j)}{dt} = -\frac{16}{45}G\epsilon_{ikl}S_{lj}^{(4)}\frac{d}{dt}\int d^3\mathbf{x}\ \rho x_kv_iv_j \\
&= \frac{16}{45}G\left(S_{ij}^{(2)}S_{ij}^{(4)} - S_{ij}^{(4)}\epsilon_{kli}\frac{d}{dt}\iint d^3\mathbf{x}\ d^3\mathbf{x}'\ \rho(\mathbf{x})\rho(\mathbf{x}')\frac{x_jx_kx'_l}{|\mathbf{x}-\mathbf{x}'|^3}\right)\ ,
\end{aligned} \tag{40}
$$

$$
\int d^3\mathbf{x}\ \rho v_iv_jv_k\partial_k({}_7h_{ij}) = 0\ . \tag{41}
$$

In deriving (38)–(41) we have made use of the Newtonian continuity and Euler equations as well as of the relation [cf. equation (76a) in Appendix A]

$$
S_{ij}^{(1)} = \int d^3\mathbf{x}\ \rho\epsilon_{kl(i}v_{j)}x_kv_l + \iint d^3\mathbf{x}\ d^3\mathbf{x}'\ \rho(\mathbf{x})\rho(\mathbf{x}')\frac{\epsilon_{kl(i}x_{j)}x_kx'_l}{|\mathbf{x}-\mathbf{x}'|^3}\ . \tag{42}
$$

Grouping all the terms, we therefore obtain

$$
\begin{aligned}
\frac{dE}{dt} &= \frac{16}{45}\frac{G}{c^7}\Big(-2S^{(1)}_{ij}S^{(5)}_{ij} - S^{(2)}_{ij}S^{(4)}_{ij} + 2S^{(5)}_{ij}\epsilon_{kli}\iint d^3\mathbf{x}\, d^3\mathbf{x}'\ \rho(\mathbf{x})\rho(\mathbf{x}')\frac{x_j x_k x'_l}{|\mathbf{x}-\mathbf{x}'|^3} \\
&\qquad + 2S^{(4)}_{ij}\epsilon_{kli}\frac{d}{dt}\iint d^3\mathbf{x}\, d^3\mathbf{x}'\ \rho(\mathbf{x})\rho(\mathbf{x}')\frac{x_j x_k x'_l}{|\mathbf{x}-\mathbf{x}'|^3}\Big) \\
&= -\frac{16}{45}\frac{G}{c^7}S^{(3)}_{ij}S^{(3)}_{ij} + \frac{16}{45}\frac{G}{c^7}\frac{d}{dt}\Big[-2S^{(1)}_{ij}S^{(4)}_{ij} + S^{(2)}_{ij}S^{(3)}_{ij} + 2S^{(4)}_{ij}\epsilon_{kli}\iint d^3\mathbf{x}\, d^3\mathbf{x}'\ \rho(\mathbf{x})\rho(\mathbf{x}')\frac{x_j x_k x'_l}{|\mathbf{x}-\mathbf{x}'|^3}\Big]\ .
\end{aligned} \tag{43}
$$

As done in Section 3, we now assume quasi-periodicity in the mass-current quadrupole moments and average expression (43) over several periods. This allows us to discard the total time derivative term and finally obtain the required result (22).

4.2. *Rate of Angular Momentum Loss*

Using equation(37), equation(14) leads to the following terms:

$$
\int d^3\mathbf{x}\ \rho\epsilon_{ijk}x_j\partial_{k\,9}\alpha = \frac{16}{45}G\iint \rho(\mathbf{x})\rho(\mathbf{x}')\Big(S^{(4)}_{bl}x_b x'_i\frac{x_l - x'_l}{|\mathbf{x}-\mathbf{x}'|^3} + \epsilon_{ijk}\epsilon_{mab}S^{(4)}_{bk}\frac{x_j x_m x'_a}{|\mathbf{x}-\mathbf{x}'|^3}\Big)\ , \tag{44}
$$

$$
\begin{aligned}
&\int d^3\mathbf{x}\ \rho\epsilon_{ijk}x_j\Big[\partial_t({}_7h_{kl}v_l) + \partial_m({}_7h_{kl}v_l)v_m\Big] = \\
&\qquad \frac{16}{45}G\Big[2\epsilon_{ijk}S_{bj}S^{(5)}_{bk} + \int d^3\mathbf{x}\,\rho\Big(|\mathbf{x}|^2 v_l S^{(5)}_{li} - x_k x_b v_i S^{(5)}_{bk} - x_b v_l v_i S^{(4)}_{bl} + x_m v_m v_l S^{(4)}_{il}\Big) \\
&\qquad\qquad + \iint \rho(\mathbf{x})\rho(\mathbf{x}')\, x_i x_b\Big(\frac{x_l - x'_l}{|\mathbf{x}-\mathbf{x}'|^3}S^{(4)}_{bl} - |\mathbf{x}|^2\frac{x_l - x'_l}{|\mathbf{x}-\mathbf{x}'|^3}S^{(4)}_{il} - \epsilon_{ijk}\epsilon_{lab}\frac{x_j x_a x'_l}{|\mathbf{x}-\mathbf{x}'|^3}S^{(4)}_{bk}\Big)\Big]\ ,
\end{aligned} \tag{45}
$$

$$
\frac{1}{2}\int d^3\mathbf{x}\ \rho(\mathbf{x})\epsilon_{ijk}x_j v_l v_m\partial_{k\,7}h_{lm} = \frac{16}{45}G\int d^3\mathbf{x}\ \rho(x_b v_i v_m S^{(4)}_{bm} - x_l v_l v_m S^{(4)}_{im})\ , \tag{46}
$$

where we used the Newtonian equation of motion to derive the right-hand side of equation (45).

Grouping all the terms, we then obtain

$$
\begin{aligned}
\frac{dS_i}{dt} &= \frac{16}{45}\frac{G}{c^7}\Big[-2\epsilon_{ijk}S_{bj}S^{(5)}_{bk} + \int d^3\mathbf{x}\ \rho\Big\{(x_k x_b v_i S^{(5)}_{bk} - |\mathbf{x}|^2 v_l S^{(5)}_{li}) + 2(x_b v_l v_i S^{(4)}_{bl} - x_m v_m v_l S^{(4)}_{li})\Big\} \\
&\qquad + \iint d^3\mathbf{x}\, d^3\mathbf{x}'\ \rho(\mathbf{x})\rho(\mathbf{x}')\Big(-\frac{x_b x_l(x_i - x'_i)}{|\mathbf{x}-\mathbf{x}'|^3}S^{(4)}_{bl} + |\mathbf{x}|^2\frac{(x_l - x'_l)}{|\mathbf{x}-\mathbf{x}'|^3}S^{(4)}_{li}\Big)\Big] \\
&= \frac{16}{45}\frac{G}{c^7}\Big[-2\epsilon_{ijk}S_{bj}S^{(5)}_{bk} + \frac{d}{dt}\int d^3\mathbf{x}\ \rho(x_k x_b v_i S^{(4)}_{bk} - |\mathbf{x}|^2 v_l S^{(4)}_{li})\Big].
\end{aligned} \tag{47}
$$

Once again, we can average equation (47) over several periods and obtain the required result (25).

5. (0 + 3.5) PN HYDRODYNAMIC EQUATIONS

In this Section we present the final set of $(0+3.5)$ PN hydrodynamical equations in which the 3.5 PN radiation-reaction forces depend *only* on a time-varying mass-current quadrupole moment. We will present them in a general gauge first and then distinguish the expressions resulting from Blanchet's gauge and from our new gauge.

The general expressions (4) and (6) for the continuity and Euler equations can be rewritten as

$$
\frac{\partial\rho}{\partial t} + \frac{\partial(\rho v_i)}{\partial x_i} = 0\ , \tag{48}
$$

$$
\frac{\partial(\rho w_k)}{\partial t} + \frac{\partial(\rho w_k v_i)}{\partial x_i} = -\frac{\partial P}{\partial x_k} - \rho\frac{\partial[\phi + \epsilon({}_9\alpha)]}{\partial x_k} + \epsilon\rho w_j\frac{\partial({}_8\beta_j)}{\partial x_k} + \epsilon\frac{\rho w_i w_j}{2}\frac{\partial({}_7h_{ij})}{\partial x_k}\ , \tag{49}
$$

where

$$
w_k \equiv cu_k = \epsilon({}_8\beta_k) + v_j[\delta_{jk} + \epsilon({}_7h_{jk})]\ , \tag{50}
$$

and where we define $\epsilon = c^{-7}$ to highlight the radiation-reaction contributions. Note that the left hand side of equation (49) contains a partial time derivatives of w_k rather than of v_k. Doing this removes the partial time derivatives of ${}_8\beta_k$ and ${}_7h_{jk}v_j$ from the right hand side [cf. equation (12)]. Moreover, since $w_j - v_j = O(\epsilon)$, all the quantities w_j on the right hand side of equation (49) can be replaced by the equivalent quantities v_j, whenever this is numerically more convenient.

In a similar way, the energy equation (5) can be rewritten as

$$\frac{\partial(\rho\varepsilon)}{\partial t}+\frac{\partial(\rho\varepsilon v_i)}{\partial x_i}=-P\frac{\partial v_i}{\partial x_i}\ , \tag{51}$$

or, if we define $\bar{E}\equiv\varepsilon+\frac{1}{2}w_k w_k$, in the equivalent form

$$\frac{\partial(\rho\bar{E})}{\partial t}+\frac{\partial(\rho\bar{E}+P)v_i}{\partial x_i}= \\ -\rho v_i\frac{\partial[\phi+\epsilon({}_9\alpha)]}{\partial x_i}-\epsilon({}_8\beta_j+{}_7h_{ij}v_i)\left(\frac{\partial P}{\partial x_j}+\rho\frac{\partial\phi}{\partial x_j}\right)+\epsilon\rho v_i v_j\frac{\partial({}_8\beta_j)}{\partial x_i}+\epsilon\frac{\rho v_i v_j v_k}{2}\frac{\partial({}_7h_{jk})}{\partial x_i}+O(\epsilon^2)\ . \tag{52}$$

In the specific case of an equation of state

$$P=(\Gamma-1)\rho\varepsilon\ , \tag{53}$$

the energy equation (52) can be written in a (third) simpler form

$$\frac{\partial e}{\partial t}+\frac{\partial(e v_i)}{\partial x_i}=0\ , \tag{54}$$

where $e=(\rho\varepsilon)^{1/\Gamma}$.

As discussed in Sections 3 and 4, the metric coefficients ${}_9\alpha$, ${}_8\beta_k$ and ${}_7h_{ij}$ appearing in (49)–(52) represent the radiation-reaction potentials and their expressions vary according to the gauge assumed. In particular, they have the form

$$\begin{array}{ll} \mathcal{N}ew\ Gauge & \mathcal{B}lanchet's\ Gauge \\ {}_9\alpha=\frac{1}{2}({}_7h_{ij})\,(x_i\partial_j\phi+\partial_j P_i)\ , & {}_9\alpha=0\ , \\ {}_8\beta_k=0\ , & {}_8\beta_k=\frac{16\,G}{45}\epsilon_{ijk}x_i x_l S^{(5)}_{jl}\ , \\ {}_7h_{ij}=-\frac{32\,G}{45}x_k\epsilon_{kl(i}S^{(4)}_{j)l}\ , & {}_7h_{ij}=0\ , \end{array} \tag{55}$$

where P_i is the solution of equation (36). We stress that the most important difference in the two gauges is given by the appearance of a fourth or of a fifth time derivative of S_{ij}. Note also that, in the new gauge, both the last term of equation (52), as well as terms including ${}_8\beta_k$, are zero.

Finally, the set of hydrodynamical equations is closed by the Poisson's equation for the Newtonian gravitational potential ϕ

$$\Delta\phi=4\pi G\rho\ . \tag{56}$$

In the case of the new gauge, equation (56) needs to be supplemented by three additional elliptic equations for the components of the vector potential P_i

$$\Delta P_i=-4\pi G\rho x_i\ . \tag{57}$$

Boundary conditions at $r\to\infty$ for the linear elliptic equations (56) and (57) are given by

$$\phi(r)=\frac{G}{r}\int d^3\mathbf{x}\ \rho+O(r^{-3})\ ,\qquad P_i=\frac{Gx_k}{r^3}\int d^3\mathbf{x}\ x_k x_i\rho+O(r^{-3})\ . \tag{58}$$

Computing the amplitude and waveforms of the gravitational waves emitted is clearly of great interest since they provide the contact with the observations and can be used to extract astrophysical information about the source. In the wave zone (Thorne 1980) and at a distance $r=|\mathbf{x}|$ from the source (where $r\gg L$, with L being the size of the source) the gravitational wave field is described by the transverse-traceless (TT) part of the linear 3-metric perturbations (Thorne 1980; Blanchet 1993, 1997).

$$(\bar{h}^{ij}_{(1)})^{\rm TT}(t,\,\mathbf{x})=-\frac{4G}{c^4}\sum_{l=2}^{\infty}\left\{\frac{(-1)^l}{l!}\frac{1}{r}\partial_{L-2}\left[M^{(2)}_{ijL-2}\left(t-\frac{r}{c}\right)\right]+2\frac{(-1)^l l}{(l+1)!}\frac{1}{r}\partial_{aL-2}\left[\epsilon_{ab(i}S^{(1)}_{j)bL-2}\left(t-\frac{r}{c}\right)\right]\right\}^{\rm TT}+O(r^{-2})\ , \tag{59}$$

where $\partial_L = \partial_{i_1}\partial_{i_2}\ldots\partial_{i_l}$ and $M_L(t-r/c)$, $S_L(t-r/c)$ are the $(L = i_1 i_2 \ldots i_l)$-th mass multipole moment and mass-current multipole moment respectively. The superscript TT refers to projecting out the transverse, traceless part:

$$[A_{ij}]^{\rm TT} \equiv P_{il}P_{jm}A_{lm} - \frac{1}{2}P_{ij}P_{lm}A_{lm} \ , \tag{60}$$

where $P_{ij} \equiv \delta_{ij} - n_i n_j$ is the projection operator and δ_{ij} the usual Kronecker-delta symbol. Restricting our attention to the contribution given by mass-current quadrupoles, equation (59) then gives

$$(\bar{h}^{ij}_{(1)})^{\rm TT} = -\frac{8G}{3c^5}\left[\epsilon_{ab(i}S^{(2)}_{j)a} - \epsilon_{ab(i}n_{j)}n_k S^{(2)}_{ka}\right]\frac{n_b}{r} \ . \tag{61}$$

Note that at the 3.5 PN order, there are no gravitational-wave "tail effects" and therefore the mass-current quadrupole S_{ij} corresponds to the gravitational wave moment observed. The usual states of polarization of the gravitational waves emitted, $h_+(\theta,\,\varphi)$ and $h_\times(\theta,\,\varphi)$ at a coordinate position $(\theta,\,\varphi)$ on the 2-sphere of radius r, can be derived from (61) after selecting the orientation of the source and thus the direction of propagation of the waves (Rasio and Shapiro 1994, Ruffert, Janka and Schäfer 1996).

6. TIMESCALES AND RESCALING

As mentioned in Section 1, we here further explore the possibility of using the set of (0 + 3.5) PN hydrodynamical equations presented above to investigate the onset and growth of the r-mode instability. In particular, we want to address the problem of the timescales and propose a strategy to suitably rescale them.

It is commonly assumed that the evolution of the r-mode instability in a unmagnetized, rotating neutron star proceeds through three stages (Owen et al. 1998). During the *initial* stage, any infinitesimal (axial) perturbation grows exponentially in a timescale $\tau_{_{\rm GR}}$, set by gravitational radiation-reaction. This is followed by the *intermediate* stage during which the amplitude of the mode saturates due to (not yet well understood) nonlinear hydrodynamic effects; the star is progressively spun-down as a result of the angular momentum loss via gravitational waves. The *final* stage of the evolution occurs when the star's angular velocity is so small that viscous dissipative effects dominate the radiation-reaction forces and the r-mode oscillations are damped out. The first stage, for a $\ell = m = 2$ mode, has been estimated to be of the order of a few seconds for a neutron star initially rotating at the break-up limit for several different equations of state, while the second to be of the order of about one year (Lindblom, Owen and Morsink 1998).

One complication in simulating r-mode oscillations is that the "natural" timescale $\tau_{_{\rm GR}}$ for the instability to grow and saturate is likely to be much longer than the timescale over which a numerical computation can be carried out. Even the most sophisticated three-dimensional Newtonian numerical codes suffer from numerical viscosity and are able to preserve accurate configurations only for a limited number of stellar rotations (i.e. $\lesssim 10-100$) and this might well be insufficient for the instability to saturate. Below we review the relevant timescales for the r-mode instability and propose a strategy whereby, with suitable scaling, we can achieve these timescales in a numerical simulation. Our brief review follows closely the results presented by Lindblom, Owen and Morsink (1998).

Perturbation analysis can be used to estimate $\tau_{_{\rm GR}}$ by assuming that the rate of energy loss to gravitational radiation emission grows according to

$$\left(\frac{d\widetilde{E}}{dt}\right)_{_{\rm GR}} = -\frac{2\widetilde{E}}{\tau_{_{\rm GR}}} \ , \tag{62}$$

where $\widetilde{E} > 0$ is the energy in the mode and $\tau_{_{\rm GR}} < 0$. In the corotating frame of the equilibrium unmagnetized star, $\widetilde{E}$ can be calculated as

$$\widetilde{E} = \frac{1}{2}\int\left[\rho\,\delta\mathbf{v}\cdot\delta\mathbf{v}^* + \left(\frac{\delta p}{\rho} - \delta\phi\right)\delta\rho^*\right]d^3\mathbf{x} \ . \tag{63}$$

The lowest order expressions for the Eulerian density perturbation $\delta\rho$ and velocity perturbations δv^a can be deduced from the perturbed fluid equations and, in a spherical coordinate system $(r,\ \theta,\ \varphi)$, have the form (Lindblom, Owen and Morsink 1998; Lindblom, Mendell and Owen 1999)

$$\frac{\delta\rho}{\rho} = \frac{(2\ell+1)}{\ell(\ell+1)\sqrt{2\ell+3}}\left(\frac{\alpha_{_A}R^2\Omega^2}{\sqrt{2\ell+3}}\right)\frac{d\rho}{dp}\left[\frac{2\ell}{2\ell+1}\sqrt{\frac{\ell}{\ell+1}}\left(\frac{r}{R}\right)^{\ell+1} + \delta\Psi(r)\right]Y_{\ell+1\,\ell}\,e^{i\omega t} \ , \tag{64}$$

where $\delta\Psi(r)$ is proportional to the gravitational potential, and the axial velocity perturbations are given by

$$\delta\mathbf{v} = \alpha_{_A}R\Omega\left(\frac{r}{R}\right)^{\ell}\mathbf{Y}^B_{\ell m}e^{i\omega t} \ . \tag{65}$$

Here, R and Ω are the radius and angular velocity of the unperturbed star, $\alpha_{_A}(t)$ is a dimensionless coefficient parameterizing the amplitude of the perturbation, ω is the (Eulerian) frequency of the mode, and $\mathbf{Y}^B_{\ell m}$ is the magnetic-type vector

spherical harmonic. Given an axial perturbation with periodic dependence $e^{i(m\varphi+\omega t)}$ and the definition (63) of the energy in the mode, the perturbed fluid equations yield the following general expression for the time derivative of $\widetilde{E}$ (Ipser and Lindblom 1991; Lindblom, Owen and Morsink 1998)

$$\frac{d\widetilde{E}}{dt} = -\int \left(2\eta\delta\sigma^{ab}\delta\sigma^*_{ab} + \zeta\delta\sigma\delta\sigma^*\right) d^3\mathbf{x} \; - \omega(\omega+m\Omega)\sum_{\ell\geq 2} N_\ell \omega^{2\ell}\left(|\delta I_{\ell m}|^2 + \frac{|\delta S_{\ell m}|^2}{c^2}\right) \,, \tag{66}$$

where η and ζ are the shear and bulk viscosities of the fluid (taken as given functions of the density and temperature) and

$$N_\ell = \frac{4\pi G}{c^{2\ell+1}}\frac{(\ell+1)(\ell+2)}{\ell(\ell-1)[(2\ell+1)!!]^2} \,. \tag{67}$$

It is easy to distinguish in expression (66) the suppressing viscous terms, driven by the perturbed shear $\delta\sigma_{ab}$ and expansion $\delta\sigma$, from the driving gravitational radiation terms, driven by the time-varying mass $\delta I_{\ell m}$ and mass-current $\delta S_{\ell m}$ multipole moments of the perturbed fluid[4]. The explicit contribution to the imaginary part of the frequency of the mode due to gravitational radiation-reaction can then be calculated as (Lindblom, Owen and Morsink 1998)

$$\frac{1}{\tau_{\rm GR}} = -\frac{1}{2\widetilde{E}}\left(\frac{d\widetilde{E}}{dt}\right)_{\rm GR} = -\frac{32\pi G\Omega^{2\ell+2}}{c^{2\ell+3}}\frac{(\ell-1)^{2\ell}}{[(2\ell+1)!!]^2}\left(\frac{\ell+2}{\ell+1}\right)^{2\ell+2}\int_0^R \rho\, r^{2\ell+2}dr \,. \tag{68}$$

where, as first pointed out by Papaloizou and Pringle (1978), we have used the following relation between the frequency of the mode and the star angular velocity

$$\omega = \omega_{\rm rot} - m\Omega = \frac{2m\Omega}{\ell(\ell+1)} - m\Omega = -\frac{(\ell-1)(\ell+2)}{\ell+1}\Omega \,. \tag{69}$$

Here $\omega_{\rm rot}$ is the angular frequency of the mode in the corotating frame and the last expression in (69) refers to the case in which $m=\ell$. Note that the contribution to the growth rate in (68) comes solely from the current multipole moments $\delta S_{\ell\,\ell}$ since we have implicitly neglected the contributions coming from the mass multipole moments $\delta I_{\ell\,\ell}$. Such an approximation is reasonable because the density perturbations are one order in Ω higher than the correspondent velocity perturbations and because the density perturbations generate gravitational radiation at a higher frequency (Lindblom, Owen and Morsink 1998).

The general expression for $\tau_{\rm GR}$ in (68) can be rewritten in a number of alternative ways, some of which are more useful within a computational context. Depending on whether the sequence of initial data is specified in terms of the ratio R/M, or in terms of the mass M, or of the angular velocity Ω, we can rewrite (68) respectively as

$$\tau_{\rm GR} = -A_\ell \frac{c^{2\ell+3}}{G}\frac{R^{2\ell+3}}{\mathcal{I}}\frac{1}{(\Omega M)^{2\ell+2}}\left(\frac{M}{R}\right)^{2\ell} M \,, \tag{70a}$$

$$= -A_\ell \frac{c^{2\ell+3}}{G^{\ell+2}}\frac{R^{2\ell+3}}{\mathcal{I}}\left(\frac{\Omega_{_K}}{\Omega}\right)^{2\ell+2}\left(\frac{R}{M}\right)^{\ell+3} M \,, \tag{70b}$$

$$= -A_\ell \left(\frac{c^2}{G}\right)^{\ell+3/2}\frac{R^{2\ell+3}}{\mathcal{I}}\left(\frac{\Omega_{_K}}{\Omega}\right)^{2\ell+1}\left(\frac{R}{M}\right)^{\ell+3/2}\frac{1}{\Omega} \,, \tag{70c}$$

where

$$A_\ell \equiv \frac{1}{24}\frac{[(2\ell+1)!!]^2}{(\ell-1)^{2\ell}}\left(\frac{\ell+1}{\ell+2}\right)^{2\ell+2} \,, \qquad\qquad \Omega_{_K} = \sqrt{\frac{GM}{R^3}} \tag{71}$$

and

$$\mathcal{I} \equiv \frac{1}{\bar\rho}\int_0^R \rho\, r^{2\ell+2}dr \,, \qquad\qquad \bar\rho \equiv \frac{3M}{4\pi R^3} \,. \tag{72}$$

In general, the integral in (72) needs to be computed numerically, but, in the case of a polytropic equation of state $P = K\rho^\Gamma$ with $\Gamma = 2$, it can be computed analytically and, in particular, (Jeffrey 1995)

[4] Note that $\omega(\omega+m\Omega) < 0$.

$$\frac{\mathcal{I}}{R^{2\ell+3}} = \frac{1}{3\pi^{2\ell+1}}\int_0^\pi \sin x\, x^{2\ell+1}dx = \frac{(2\ell+1)!}{3\pi^{2\ell+1}}\left\{\sum_{k=0}^{\ell}(-1)^{k+2}\frac{\pi^{2\ell-2k+1}}{(2\ell-2k+1)!}\right\} . \tag{73}$$

Suppose we now impose the condition that the *computational timescale* $\tau_{_C}$, expressed as a multiple N of the stellar rotations, be identical to the growth-time $\tau_{\rm GR}$

$$\tau_{_C} \equiv N\frac{2\pi}{\Omega} = |\tau_{\rm GR}| \tag{74}$$

Using (73) and (70c), expression (74) then becomes a condition on the ratio $c^2R/(GM)$, i.e.

$$\frac{c^2R}{GM} = \left[\frac{2\pi N}{A_\ell}\frac{\mathcal{I}}{R^{2\ell+3}}\left(\frac{\Omega}{\Omega_{_K}}\right)^{2\ell+1}\right]^{2/(2\ell+3)} \simeq 0.68\; N^{2/7} , \tag{75}$$

where the numerical coefficient comes from considering $\ell=2$, $\Gamma=2$ and $\Omega_{_K}=\Omega$. According to (75), it is always possible to rescale the value of the constant c in such a way as to make the growth-time compatible with the timescale over which the numerical computations can be carried out. Provided we maintain the inequality $\tau_{_C} = |\tau_{\rm GR}| \gg \Omega^{-1}$ or $N \gg 1$, this rescaling should in no way affect the profiles of the physical parameters during the evolution, but only shorten the evolution time over which their growth and saturation occur. Alternatively, one can choose M/R to be sufficiently large to reduce the growth-time in accord with (70b) and then scale the results to stars with more realistic compaction ratios. A similar rescaling technique has also been adopted to accelerate the cooling of a hot neutron star and study its collapse to a black hole (Baumgarte, Shapiro and Teukolsky 1996).

7. CONCLUSIONS

We have presented a new set of $(0+3.5)$ PN hydrodynamical equations in which a 3.5 PN radiation-reaction force due to a time-varying mass-current quadrupole moment is considered. Within this system of equations, the hydrodynamics is essentially Newtonian except for the inclusion of the relativistic nonconservative effects related to the emission of gravitational radiation.

We have cast this set of equations in a form which is suitable for numerical implementation. In the alternative 3.5 PN approach by Blanchet (1993, 1997), the radiation-reaction terms depend on the fifth time derivative of the mass-current quadrupole moment S_{ij}. Evaluating such a term accurately within a standard second order numerical scheme could pose a problem. Instead, we have chosen a particular gauge in which the radiation-reaction effects depend at most on the fourth time derivative of S_{ij} and can therefore be calculated accurately. The additional complication that arises with this gauge choice are three linear, elliptic equations for the components of a vector potential. The solution of such equations is no more difficult than the solution of Poisson's equation for the Newtonian gravitational potential and can be performed in an identical fashion.

Simulating the onset and growth of the r-mode instability in rotating neutron stars is highly desirable. A $(0+3.5)$PN approach may be considerably simpler than a fully general relativistic one and allows one to neglect all conservative relativistic effects, which should be perturbative, and focus exclusively on the radiation-reaction effects due to a time-varying mass-current quadrupole moment. We have also proposed a suitable rescaling that will make the timescale for the onset and saturation of the r-mode instability compatible with any reasonable integration time imposed by computational constraints. Work is presently in progress to implement these equations in a numerical code (Ruffert et al. 1999).

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APPENDIX A

STRATEGY FOR COMPUTATION OF $S^{(N)}_{IJ}$

In this Section we present the analytic integral expressions for the first, the second, and the third time derivative of the mass current quadrupole moment S_{ij}. The expressions derived here can then be used, after taking numerical time derivatives, to compute $S^{(4)}_{ij}$ and (if necessary) $S^{(5)}_{ij}$.

The first time derivative of S_{ij} is easily derived from (18), after setting $v_i = dx_i/dt$, to yield

$$S^{(1)}_{ij} = \epsilon_{kl(i}\int d^3\mathbf{x}\,\rho\left(v_{j)}v_l + x_{j)}\frac{dv_l}{dt}\right)x_k + O(\epsilon) ; \tag{76a}$$

$$= \epsilon_{kl(i}\int d^3\mathbf{x}\,\rho\left(v_{j)}v_l - x_{j)}\partial_l\phi\right)x_k + O(\epsilon) . \tag{76b}$$

In deriving (76b) we have used the continuity and the Newtonian Euler equation,

$$\frac{\partial v_i}{\partial t}+v_j\frac{\partial v_i}{\partial x_j}=-\frac{1}{\rho}\frac{\partial P}{\partial x_i}-\frac{\partial \phi}{\partial x_i}\ , \tag{77}$$

and exploited the following identity

$$\epsilon_{km(i}\int d^3\mathbf{x}\, x_{j)}x_k\partial_m P=0\ . \tag{78}$$

A similar procedure is used for the second time derivative, which can be written as

$$\begin{aligned} S^{(2)}_{ij} &= \epsilon_{kl(i}\Bigg\{\int d^3\mathbf{x}\, Px_k\left[\partial_{j)}v_l+\partial_l v_{j)}\right]-\int d^3\mathbf{x}\,\rho x_k\left[(\partial_{j)}\phi)v_l+2v_{j)}\partial_l\phi\right] \\ &\qquad -\int d^3\mathbf{x}\,\rho\left[x_{j)}v_k\partial_l\phi+x_{j)}x_k\left[\partial_l(\partial_t\phi)+v_m\partial_{lm}\phi\right]\right]\Bigg\}+O(\epsilon)\ ; \qquad (79a) \\ &= \epsilon_{kl(i}\Bigg\{\int d^3\mathbf{x}\, Px_k\left[\partial_{j)}v_l+\partial_l v_{j)}\right]+\int d^3\mathbf{x}\,\nabla\cdot(\rho\mathbf{v})\left[x_{j)}x_k\partial_l\phi\right] \\ &\qquad -\int d^3\mathbf{x}\,\rho x_k\left[x_{j)}\partial_l(\partial_t\phi)+v_l\partial_{j)}\phi+v_{j)}\partial_l\phi\right]\Bigg\}+O(\epsilon)\ . \qquad (79b) \end{aligned}$$

Note that we have proposed two different expressions for $S^{(2)}_{ij}$, where the second one [i.e. (79b)] made use of the following identity

$$\begin{aligned} \int d^3\mathbf{x}\,\rho\left[v_{(i}\partial_{j)}\phi+v_k x_{(i}\partial_{j)k}\phi\right] &= \int d^3\mathbf{x}\,\rho v_k\partial_k\left[x_{(i}\partial_{j)}\phi\right]=\int d^3\mathbf{x}\,\partial_k\left[\rho v_k x_{(i}\partial_{j)}\phi\right]-\int d^3\mathbf{x}\,\left[x_{(i}\partial_{j)}\phi\right]\nabla\cdot(\rho\mathbf{v}) \\ &= -\int d^3\mathbf{x}\,\left[x_{(i}\partial_{j)}\phi\right]\nabla\cdot(\rho\mathbf{v})\ , \end{aligned} \tag{80}$$

in order to eliminate the mixed second partial derivatives of the gravitational potential.

Expressions (76a)–(79b) apply for a generic equation of state. However, in deriving $S^{(3)}_{ij}$ we will need a time derivative of the volume integral of the pressure and a specific equation of state must be specified. In the case of an equation of state $P=(\Gamma-1)\rho\varepsilon$, we obtain

$$\begin{aligned} S^{(3)}_{ij} = \epsilon_{kl(i}\Bigg\{ &\int d^3\mathbf{x}\,(\Gamma-1)\rho\varepsilon\Big\{(1-\Gamma)x_k\left(\partial_{j)}v_l+\partial_l v_{j)}\right)(\partial_m v_m)+v_k\left(\partial_{j)}v_l+\partial_l v_{j)}\right) \\ &\qquad +x_k\left[\partial_{j)}a_l+\partial_l a_{j)}-(\partial_{j)}v_n)(\partial_n v_l)-(\partial_l v_n)(\partial_n v_{j)})\right]\Big\} \\ &-\int d^3\mathbf{x}\,\rho\left\{x_{j)}a_k\partial_l\phi+3v_{j)}v_k\partial_l\phi+x_k\left[a_l\partial_{j)}\phi+2a_{j)}\partial_l\phi+v_l\partial_{j)}(\partial_t\phi)+v_lv_n\partial_{j)n}\phi\right]\right\} \\ &-\int d^3\mathbf{x}\,\rho\left\{[3v_{j)}x_k+2x_{j)}v_k][\partial_l(\partial_t\phi)+v_n\partial_{ln}\phi]+x_{j)}x_k\left[\partial_l\partial_{tt}\phi+2v_n\partial_{ln}(\partial_t\phi)+a_n\partial_{ln}\phi\right]\right\} \\ &+\int d^3\mathbf{x}\,x_{j)}x_k\partial_l(\rho v_m v_n)\partial_{mn}\phi\Bigg]\Bigg\}+O(\epsilon)\ , \end{aligned} \tag{81a}$$

or equally

$$
\begin{aligned}
S^{(3)}_{ij} = \epsilon_{kl(i}\Bigg\{ & \int d^3\mathbf{x}\, P \left[\partial_{j)} v_l + \partial_l v_{j)}\right] (v_k - (\Gamma - 1) x_k \partial_n v_n) \\
& + \int d^3\mathbf{x}\, P x_k \left[\partial_{j)} a_l + \partial_l a_{j)} - (\partial_{j)} v_n)(\partial_n v_l) - (\partial_l v_n)(\partial_n v_{j)})\right] \\
& + \int d^3\mathbf{x}\, \nabla\cdot(\rho\mathbf{v}) \Big[\partial_l \phi \left(v_{j)} x_k + x_{j)} v_k\right) + x_{j)} x_k \left[\partial_l(\partial_t\phi) + v_n \partial_{nl}\phi\right]\Big] \\
& + \int d^3\mathbf{x}\, x_{j)} x_k \partial_l \phi \left[\partial_n(\rho a_n) - \partial_n(\rho v_m \partial_m v_n)\right] - \int d^3\mathbf{x}\, \rho v_k \left[x_{j)} \partial_l(\partial_t\phi) + \left(\partial_{j)}\phi\right) v_l + v_{j)} \partial_l \phi\right] \\
& - \int d^3\mathbf{x}\, \rho x_k \Big[v_{j)} \left[2\partial_l(\partial_t\phi) + v_n \partial_{nl}\phi\right] + x_{j)} \left[\partial_l(\partial_{tt}\phi) + v_n \partial_{nl}(\partial_t\phi)\right] \\
& \qquad v_l + \left[\partial_{j)}(\partial_t\phi) + v_n \partial_{nj)}\phi\right] + a_l \partial_{j)}\phi + a_{j)} \partial_l \phi \Big] \Bigg\} .
\end{aligned} \tag{81b}
$$

where we have defined

$$
a_i \equiv \frac{dv_i}{dt} = -\frac{1}{\rho}\partial_i P - \partial_i \phi , \tag{82}
$$

and used the relation

$$
\frac{d}{dt}\int d^3\mathbf{x}\, P = \int d^3\mathbf{x}\, \rho \frac{d}{dt}\left(\frac{P}{\rho}\right) = -\int d^3\mathbf{x}\, P(\Gamma - 1)\partial_j v_j . \tag{83}
$$

The partial time derivatives of the gravitational potential $(\partial_t\phi)$ and $(\partial_{tt}\phi)$ appearing in (79) and (81) satisfy the following elliptic equations (Nakamura and Oohara 1989)

$$
\Delta(\partial_t\phi) = -4\pi G \partial_k(\rho v_k) , \tag{84}
$$

$$
\Delta(\partial_{tt}\phi) = 4\pi G[\partial_{ij}(\rho v_i v_j) + \Delta P + \partial_i(\rho \partial_i \phi)] , \tag{85}
$$

where Δ denotes the flat spatial Laplacian. While $S^{(4)}_{ij}$ and $S^{(5)}_{ij}$ could also be expressed through similar integral expressions, this is not useful in general. For a numerical method which is accurate in time and space at the order n, the maximum time derivative of S_{ij} which can be calculated accurately is $(n+1)$. This is because, for a generic $S^{(n+1)}_{ij}$, we need n spatial derivatives and n partial time derivatives of ϕ. As a result, if one is using a numerical method which is second order accurate in space and time, analytic integral expressions are reliable at most up to $S^{(3)}_{ij}$. The fourth and fifth time derivatives need to be obtained by finite differencing of $S^{(3)}_{ij}$ with increasingly larger truncation errors. This consideration is the guideline for our formalism, for which we need only compute $S^{(4)}_{ij}$.

APPENDIX B

CANONICAL FORM OF THE LINEAR METRIC

Here we obtain the parts of the metric associated with radiation-reaction potential. We follow the notation of Blanchet (1993). We first recall that the components of the linearized metric $\tilde{h}^{\mu\nu}_{(1)}$ in canonical form and in the harmonic gauge condition (Thorne 1980) are given by

$$
\tilde{\alpha} = 0 , \tag{86}
$$

$$
\tilde{\beta}^i = \frac{4G}{c^3}\sum_{l=1}^{\infty}\frac{(-1)^l l}{(l+1)!}\epsilon_{iab}\partial_{aL-1}\left[\frac{1}{r}S_{bL-1}\Big(t - \frac{r}{c}\Big)\right] , \tag{87}
$$

$$
\tilde{h}_{ij} = \frac{8G}{c^4}\sum_{l=2}^{\infty}\frac{(-1)^l l}{(l+1)!}\partial_{aL-2}\left[\frac{1}{r}\epsilon_{ab(i}S^{(1)}_{j)bL-2}\Big(t - \frac{r}{c}\Big)\right] , \tag{88}
$$

where we have considered *only* the terms related to mass-current multipole moments S_L. Here $\partial_L = \partial_{i_1}\partial_{i_2}\ldots\partial_{i_l}$ and $L = i_1 i_2 \ldots i_l$ is a compact expression for l indices. Since there is no "tail" term at the linear order, we can derive the radiation-reaction metric by taking the half-difference of the retarded and advanced waves (Blanchet 1993)

$$\tilde{\beta}^i = \frac{4G}{c^3}\sum_{l=1}^{\infty}\frac{(-1)^l l}{(l+1)!}\epsilon_{iab}\partial_{aL-1}\left[\frac{S_{bL-1}(t-r/c) - S_{bL-1}(t+r/c)}{2r}\right] , \tag{89}$$

$$\tilde{h}_{ij} = \frac{8G}{c^4}\sum_{l=2}^{\infty}\frac{(-1)^l l}{(l+1)!}\partial_{aL-2}\left[\epsilon_{ab(i}\frac{S^{(1)}_{j)bL-2}(t-r/c) - S^{(1)}_{j)bL-2}(t+r/c)}{2r}\right] . \tag{90}$$

Expanding the numerators of the right-hand sides with respect to c^{-1} in order to determine the near zone metric, we obtain equations (26a) and (26b) as the lowest order of the $l = 2$ mode. The gauge transformation necessary in order to set the "new" linear shift $\beta^i = 0$ is therefore simply given by

$$\xi^i = -\frac{4G}{c^2}\sum_{l=2}^{\infty}\frac{(-1)^l l}{(l+1)!}\epsilon_{iab}\partial_{aL-1}\left[\frac{S^{(-1)}_{bL-1}(t-r/c) - S^{(-1)}_{bL-1}(t+r/c)}{2r}\right] , \tag{91}$$

Using (91), the expression of $\tilde{h}_{ij}$ in the new gauge is

$$\begin{aligned} h_{ij} = \tilde{h}_{ij} + 2\partial_{(i}\xi_{j)} = &\frac{8G}{c^4}\sum_{l=2}^{\infty}\frac{(-1)^l l}{(l+1)!}\frac{l+2}{2l+1}\partial_{aL-2}\left[\epsilon_{ab(i}\frac{S^{(1)}_{j)bL-2}(t-r/c) - S^{(1)}_{j)bL-2}(t+r/c)}{2r}\right] \\ &-\frac{8G}{c^2}\sum_{l=2}^{\infty}\frac{(-1)^l l}{(l+1)!}\epsilon_{ab(i}\hat{\partial}_{j)aL-1}\left[\frac{S^{(-1)}_{bL-1}(t-r/c) - S^{(-1)}_{bL-1}(t+r/c)}{2r}\right] , \end{aligned} \tag{92}$$

where $\hat{\partial}_L$ is the (symmetric) trace-free part of ∂_L. Equation (92) should be compared with the equivalent one obtained in Blanchet's gauge (we recall that we report here only the contributions due to time-varying mass-current multipoles) [cf. equation (2.8c) of Blanchet 1997]

$$\left(h_{ij}\right)^{\text{Blanchet's}}_{\text{gauge}} = -\frac{8G}{c^2}\sum_{l=2}^{\infty}\frac{(-1)^l l}{(l+1)!}\frac{2l+1}{l-1}\epsilon_{ab(i}\hat{\partial}_{j)aL-1}\left[\frac{S^{(-1)}_{bL-1}(t-r/c) - S^{(-1)}_{bL-1}(t+r/c)}{2r}\right] . \tag{93}$$

Expression (93), as well as the the second term in (92), provide no contribution at 3.5 PN order.

Using (89) and (92) at the lowest order of the PN expansion of the $l = 2$ mode, we obtain equations (29).

APPENDIX C

TIME-SLICE CONDITION AND EQUATION FOR $_9\alpha$

We here discuss the choice of a time-slice condition and the derivation of the elliptic equation (30) for $_9\alpha$. We start by considering the evolution equation for the 3-metric γ_{ij}

$$\frac{1}{c}\partial_t\gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i , \tag{94}$$

and the momentum constraint equations

$$D_j K^j_{\ i} - D_i K = \frac{8\pi G}{c^4}J_i , \tag{95}$$

where D_i is the covariant derivative with respect to γ_{ij}, K_{ij} is the extrinsic curvature tensor, and $K = K_i^{\ i}$. The current source term on the left hand side of (95) is defined as $J_\mu \equiv -\gamma_{\mu\nu}T^{\nu\alpha}n_\alpha$, with $n^\mu = (1/\alpha, -\beta^k/\alpha)$ being the normal to the spatial slice. The 3.5 PN expressions of (94) and (95) are given respectively by

$$\frac{1}{c}\partial_t({}_7h_{ij}) = -2{}_8K_{ij} + \partial_i({}_8\beta_j) + \partial_j({}_8\beta_i) , \tag{96}$$

and

$$\partial_i({}_8K_{ij}) - \partial_j({}_8K) = 0 \ , \tag{97}$$

where ${}_4J_i = 0$ (Asada, Shibata, and Futamase 1996).

Taking a further spatial derivative of (96) and using the constraints (97), we then obtain

$$2\partial_i({}_8K) = 2\partial_i({}_8K_{ij}) = \partial_{ii}({}_8\beta_j) + \partial_{ij}({}_8\beta_i) - \frac{1}{c}\partial_{ti}({}_7h_{ij}) \ . \tag{98}$$

Performing now an infinitesimal gauge transformation yielding [cf. equations (27)–(29b)]

$${}_8\beta_i = 0 \ , \tag{99}$$

$${}_7h_{ij} = -\frac{32\,G}{45}x_k\epsilon_{kl(i}S^{(4)}_{j)l} \ . \tag{100}$$

will set to zero the right hand side of equation (98) [we recall that $\partial_i({}_7h_{ij}) = 0$] and thus require ${}_8K$ to be a constant. As a result, we choose as slice condition

$$K = 0 \ , \tag{101}$$

at all times. Condition (101) is known as the maximal time-slice condition (Arnowitt, Deser and Misner 1962; Smarr and York 1978; Schäfer 1983; Blanchet, Damour and Schäfer 1990). As a consequence of (101), the evolution equation of K is given by

$$\frac{1}{c}\partial_t K = -D_kD^k\alpha + \frac{4\pi G}{c^2}\alpha(\rho_{_E} + S) + \alpha K_{ij}K^{ij} = 0 \ , \tag{102}$$

where $\rho_{_E} \equiv T_{\mu\nu}n^\mu n^\nu$ and $S \equiv T_{\mu\nu}\gamma^{\mu\nu}$.

The metric coefficient ${}_9\alpha$ is then obtained as the solution of the 3.5 PN expression of the elliptic equation (102):

$$D_kD^k\alpha = \frac{4\pi G}{c^2}\alpha(\rho_{_E} + S) + \alpha K_{ij}K^{ij} \ . \tag{103}$$

After discarding all the 3.5 PN terms except those arising from the mass-current quadrupole, the 3.5 PN expression of the left-hand side of equation (103) is written as [cf. equation (8)]

$$D_kD^k\alpha = \Delta({}_9\alpha) - \partial_i({}_7h_{ij}\partial_j\phi) \ , \tag{104}$$

while the (full) right-hand side of equation (103) is rewritten as

$$\frac{4\pi G}{c^2}\alpha(\rho_{_E} + S) + \alpha K_{ij}K^{ij} = \frac{4\pi G\alpha}{c^2}\left[\rho\left\{1 + \frac{1}{c^2}\left(\varepsilon + \frac{P}{\rho}\right)\right\}\left(1 + \frac{2}{c^2}\gamma^{ij}w_iw_j\right) + \frac{2}{c^2}P\right] + \alpha K_{ij}K^{ij} \ . \tag{105}$$

It is easy to estimate that, at the 3.5 PN level, the contributions to expression (105) coming from the mass current quadrupole moment are at most $O(c^{-11})$ and that the slicing condition for ${}_9\alpha$ is therefore given by (30). This follows from the fact that the contribution in ρ, ε, and P is $O(1)$, and is $O(c^{-7})$ for $\gamma^{ij}w_iw_j$. As a consequence, the contribution of the mass current quadrupole moment from the terms in the curly brackets of (105) is at most of $O(c^{-9})$. Similar considerations apply also for the last term in the right-hand side of (105) where $K_{ij} = O(c^{-3})$, while the contribution of the mass-current quadrupole moment in K_{ij} is $O(c^{-8})$ [cf. equation (94)]. As a result, the mass-current contribution in $\alpha K_{ij}K^{ij}$ is at most $O(c^{-11})$.

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